On the Invariants of a Homogeneous Quadratic Differential Equation of the Second Order.

By D. R. Curtiss.

The subject of invariants of linear differential equations under the transformation

$$\xi = \mu(x), \quad y = \lambda(x)\eta, \tag{1}$$

where $\mu(x)$ and $\lambda(x)$ are arbitrary functions of x, has been discussed by many writers. A brief summary of this work may be found in a paper by Dr. Bouton on the invariants of the general linear differential equation.* These considerations have been extended to systems of linear differential equations by Dr. Wilczynski,† at whose suggestion the work of this paper was undertaken. But although Staeckel has shown‡ that (1) is the most general point transformation which converts a homogeneous differential equation of any degree and of order greater than one into another of the same degree and order, equations of degree higher than unity have so far received very little notice.

In this paper I propose to treat the equation

$$\left(\frac{d^2y}{dx^2}\right)^2 + 4p_2\left(\frac{dy}{dx}\right)^2 + p_4 y^2 + 4p_3 y\frac{dy}{dx} + 2q_2 y\frac{d^2y}{dx^2} + 4p_1\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} = 0, \tag{2}$$

where p_1, p_2, p_3, p_4, q_2 are functions of x, and determine those functions of the coefficients and of their derivatives which are the same for (2) and for any equation obtained from (2) by any transformation of form (1). A few applications will also receive brief notice.

Appell has published a paper§ in which he finds some of the invariants

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* American Journal of Mathematics, Vol. XXI, No. 2, 1899.

† Transactions of the American Mathematical Society, Vol. II, No. 1.

‡ Crelle's Journal, Vol. CXI.

§ Journal de Mathématique, 4th Series, Vol. V.

of this equation, but he investigates only so much of the subject as is useful in certain applications which form the bulk of his paper. His work will be referred to later on.

The transformations (1) evidently form an infinite continuous group which may be defined by differential equations. We may therefore use Lie's theory of such groups, and this method of treatment constitutes the main advance of this paper over Appell's on the same subject.

A.—*Seminvariants.*

Let us first transform the dependent variable alone. Functions of the coefficients and of their derivatives which remain invariant under this transformation we shall call seminvariants.

Denoting derivatives by accents, we have

$$\left.\begin{aligned} y &= \lambda(x)\,\eta, \\ y' &= \lambda(x)\,\eta' + \lambda'(x)\,\eta, \\ y'' &= \lambda(x)\,\eta'' + 2\lambda'(x)\,\eta' + \lambda''(x)\,\eta. \end{aligned}\right\} \tag{3}$$

As Appell remarks, y, y', y'' are expressed linearly in terms of η, η', η'', so that the invariants of the ternary quadratic form corresponding to (2) will be included among the seminvariants of (2), and will, in fact, be relative invariants under the general transformation, since this again transforms y, y', y'' linearly in terms of η, η', η''.

Substituting (3) in (2), we obtain the equation

$$\eta''^2 + 4\pi_2\eta'^2 + \pi_4\eta^2 + 4\pi_3\eta'\eta + 2\rho_2\eta''\eta + 4\pi_1\eta''\eta' = 0,$$

in which the coefficients have the values

$$\left.\begin{aligned} \pi_1 &= \frac{\lambda'}{\lambda} + p_1, \quad \pi_2 = \left(\frac{\lambda'}{\lambda}\right)^2 + 2p_1\frac{\lambda'}{\lambda} + p_2, \\ \rho_2 &= \frac{\lambda''}{\lambda} + 2p_1\frac{\lambda'}{\lambda} + q_2, \\ \pi_3 &= \frac{\lambda'}{\lambda}\cdot\frac{\lambda''}{\lambda} + p_1\left\{\frac{\lambda''}{\lambda} + 2\left(\frac{\lambda'}{\lambda}\right)^2\right\} + (2p_2 + q_2)\frac{\lambda'}{\lambda} + p_3, \\ \pi_4 &= \left(\frac{\lambda''}{\lambda}\right)^2 + 4p_1\frac{\lambda'}{\lambda}\cdot\frac{\lambda''}{\lambda} + 4p_2\left(\frac{\lambda'}{\lambda}\right)^2 + 2q_2\frac{\lambda''}{\lambda} + 4p_3\frac{\lambda'}{\lambda} + p_4. \end{aligned}\right\} \tag{4}$$

The infinitesimal transformations of the group (3) are obtained by putting

$$\left.\begin{aligned}\lambda(x) &= 1 + \phi(x)\,\delta t, \\ \lambda' &= \phi'\,\delta t, \text{ etc.}\end{aligned}\right\} \tag{5}$$

$\phi(x)$ being an arbitrary function of x alone, and δt an infinitesimal.

By inserting in (4) the special values (5) for λ, λ', etc., we find the infinitesimal transformations of the coefficients to be

$$\left.\begin{aligned}&\delta p_1 = \phi'\,\delta t, \qquad \delta p_2 = 2p_1\phi'\,\delta t, \\ &\delta q_2 = [\phi'' + 2p_1\phi']\,\delta t, \\ &\delta p_3 = [p_1\phi'' + (2p_2 + q_2)\phi']\,\delta t, \\ &\delta p_4 = 2\,[q_2\phi'' + 2p_3\phi']\,\delta t.\end{aligned}\right\} \tag{6}$$

To obtain $\delta p_1'$, $\delta p_2'$, etc., $\delta p_1''$, $\delta p_2''$ etc., we need only take derivatives, of the corresponding order, of δp_1, δp_2, etc., since the independent variable is left untransformed by the group (3).

We have then, finally, the extended transformation

$$X(f)\,\delta t \equiv \sum_{i=1}^{4}\left(\frac{\partial f}{\partial p_i}\,\delta p_i + \frac{\partial f}{\partial p_i'}\,\delta p_i' + \cdots\right) + \frac{\partial f}{\partial q_2}\,\delta q_2 + \frac{\partial f}{\partial q_2'}\,\delta q_2' + \cdots \tag{7}$$

If $\qquad F(p_1, p_2, q_2, \ldots, p_1', p_2', q_2', \ldots, p_1^{(\nu)}, p_2^{(\nu)}, q_2^{(\nu)}, \ldots)$

is a seminvariant, it must be a solution of $X(f) = 0$, and conversely, any solution of this equation is a seminvariant. We shall first look for the rational integral seminvariants; we shall find that all others can be expressed in terms of these.

Let us assign to y, y', y'' the weights 0, 1, 2 respectively. Then, in order that (2) may have all of its terms of the same weight, namely 4, we must assign to p_κ the weight κ, and to q_2 the weight 2. Further, let the weight of $p_i^{(\nu)}$ be $i + \nu$, and let that of $q_2^{(\mu)}$ be $2 + \mu$, and let the weight of a product be equal to the sum of the weights of its factors. We say that an expression is isobaric of weight λ if all of its terms are of weight λ.

It is evident that there is no integral rational seminvariant of weight 1.

Seminvariants involving expressions of weight no greater than 2 must satisfy

the equation

$$\frac{\partial f}{\partial p_1}\phi' + \frac{\partial f}{\partial p_1'}\phi'' + \frac{\partial f}{\partial p_2}2p_1\phi' + \frac{\partial f}{\partial q_2}(\phi'' + 2p_1\phi') = 0\,.$$

Since ϕ is arbitrary, this breaks up into the two equations

$$\frac{\partial f}{\partial p_1} + 2p_1\frac{\partial f}{\partial p_2} + 2p_1\frac{\partial f}{\partial q_2} = 0, \quad \frac{\partial f}{\partial p_1'} + \frac{\partial f}{\partial q_2} = 0.$$

These are independent and form a complete system; hence we have two independent solutions, and these are easily found to be

$$P_2 = p_2 - p_1^2, \tag{8}$$
$$Q_2 = q_2 - p_1' - p_1^2. \tag{9}$$

Let P_3 be a seminvariant of weight 3. The most general expression isobaric of weight 3, written with undetermined constant coefficients, is

$$P_3 = ap_3 + bp_2p_1 + cq_2p_1 + dp_1^3 + ep_2' + fq_2' + gp_1'p_1 + hp_1''.$$

Hence, if this is a seminvariant, we must have

$$\begin{aligned}\frac{\delta P_3}{\delta t} = ap_1\phi'' &+ a(2p_2 + q_2)\phi' + bp_2\phi' + 2bp_1^2\phi' + cq_2\phi' + cp_1(\phi'' + 2p_1\phi') \\ &+ 3dp_1^2\phi' + 2ep_1\phi'' + 2ep_1'\phi' + f\phi''' + 2fp_1\phi'' + 2fp_1'\phi' \\ &+ gp_1'\phi' + gp_1\phi'' + h\phi''' = 0.\end{aligned}$$

From this condition follows the system of equations

$$\begin{array}{lll} f + h = 0, & a + c + 2e + 2f + g = 0, & 2a + b \qquad = 0, \\ a + c = 0, & 2b + 2c + 3d \qquad\quad = 0, & 2e + 2f + g = 0. \end{array}$$

Of these only five are independent, so that we have three independent seminvariants of weight 3, one being

$$P_3 = p_3 - 2p_2p_1 - q_2p_1 + 2p_1^3. \tag{10}$$

The others turn out to be P_2' and Q_2'. In fact, since the independent variable is untransformed, any derivative of a seminvariant will itself be a seminvariant.

P_4 must be a solution of the four equations

$$\left.\begin{aligned}
&\frac{\partial f}{\partial p_1} + 2p_1\frac{\partial f}{\partial p_2} + 2p_1'\frac{\partial f}{\partial p_2'} + 2p_1''\frac{\partial f}{\partial p_2''} + 2p_1\frac{\partial f}{\partial q_2} + 2p_1'\frac{\partial f}{\partial q_2'} \\
&\qquad + 2p_1''\frac{\partial f}{\partial q_2''} + (2p_2+q_2)\frac{\partial f}{\partial p_3} + (2p_2'+q_2')\frac{\partial f}{\partial p_3'} + 4p_3\frac{\partial f}{\partial p_4} = 0, \\
&\frac{\partial f}{\partial p_1'} + 2p_1\frac{\partial f}{\partial p_2'} + 4p_1'\frac{\partial f}{\partial p_2''} + \frac{\partial f}{\partial q_2} + 2p_1\frac{\partial f}{\partial q_2'} + 4p_1'\frac{\partial f}{\partial q_2''} + p_1\frac{\partial f}{\partial p_3} \\
&\qquad + p_1'\frac{\partial f}{\partial p_3'} + (2p_2+q_2)\frac{\partial f}{\partial p_3'} + 2q_2\frac{\partial f}{\partial p_4} = 0, \\
&\frac{\partial f}{\partial p_1''} + 2p_1\frac{\partial f}{\partial p_2''} + \frac{\partial f}{\partial q_2'} + 2p_1\frac{\partial f}{\partial q_2''} = 0, \\
&\frac{\partial f}{\partial p_1'''} + \frac{\partial f}{\partial q_2''} = 0.
\end{aligned}\right\} \quad (11)$$

Without going through the details of solving this set, we may notice that it is a complete system of four independent equations containing thirteen arguments; the nine independent solutions are P_2, P_2', P_2'', Q_2, Q_2', Q_2'', P_3, P_3' and

$$P_4 = p_4 - 4p_3p_1 + 4p_2p_1^2 + 2q_2p_1^2 - 3p_1^4 - 2q_2p_1' + p_1'^2 + 2p_1'p_1^2. \quad (12)$$

Turning now to the general case, and considering the case of irrational as well as of rational seminvariants, it is evident that any function of seminvariants alone is itself a seminvariant. The question now arises: Can we obtain a complete system of seminvariants, i. e., a set such that all other seminvariants are functionally dependent upon it? We can answer this in the affirmative; in fact, P_2, Q_2, P_3, P_4 and their successive derivatives constitute such a set. For the fundamental differential equation, taken to terms of weight ν, will contain the first ν derivatives of ϕ; we shall have then a complete system of ν equations in the $5\nu - 7$ arguments

$$\left.\begin{matrix}
p_1, p_1', \;\ldots\ldots, p_1^{(\nu-1)}, \\
p_2, p_2', \;\ldots\ldots, p_2^{(\nu-2)}, \\
q_2, q_2', \;\ldots\ldots, q_2^{(\nu-2)}, \\
p_3, p_3', \;\ldots\ldots, p_3^{(\nu-3)}, \\
p_4, p_4', \;\ldots\ldots, p_4^{(\nu-4)}.
\end{matrix}\right\} \quad (13)$$

That these ν equations are all independent, follows at once from the fact, illustrated by (11), for the case $\nu = 4$, that each contains one and only one term of the form $\frac{\partial f}{\partial p_1^{(r)}}$, r being different for each equation. Accordingly, there

are $4\nu - 7$ seminvariants, and no more, functionally independent, involving the arguments (13) only. Such a set is formed by the quantities

$$\left.\begin{array}{l} P_2,\ P_2',\ \ldots\ldots,\ P_2^{(\nu-2)}, \\ Q_2,\ Q_2',\ \ldots\ldots,\ Q_2^{(\nu-2)}, \\ P_3,\ P_3',\ \ldots\ldots,\ P_3^{(\nu-3)}, \\ P_4,\ P_4',\ \ldots\ldots,\ P_4^{(\nu-4)}; \end{array}\right\} \qquad (14)$$

for they involve only the arguments (13), are $4\nu - 7$ in number, and are all independent. The truth of this last statement may be shown thus: The members of any row of (14) are evidently independent of each other; the second row has in each member a term of the form $q_2^{(r)}$ not to be found in the first row; the third row has in each member a term of the form $p_3^{(r)}$ not to be found in the two preceding rows; the last row, a term of the form $p_4^{(r)}$ not to be found in any preceding row.

We have thus demonstrated

THEOREM I.—*All seminvariants, rational or otherwise, of equation* (2) *are functionally dependent on* P_2, Q_2, P_3, P_4 *and their successive derivatives.*

B.—*Invariants.*

It is evident that invariants can be functions only of the seminvariants. If we apply to (2) the transformation $\xi = \mu(x)$, we need only examine how this affects the seminvariants, obtaining in terms of them the differential equation an invariant must satisfy. It should be noted that the derivative of an invariant is not, for this general transformation, an invariant, since the independent variable is now also transformed.

We have

$$\left.\begin{array}{l} \xi = \mu(x), \\ \dfrac{dy}{dx} = \dfrac{dy}{d\xi}\,\mu', \\ \dfrac{d^2y}{dx^2} = \dfrac{d^2y}{d\xi^2}\,(\mu')^2 + \dfrac{dy}{d\xi}\,\mu''. \end{array}\right\} \qquad (15)$$

Equation (2) now becomes

$$\left(\frac{d^2y}{d\xi^2}\right)^2 + 4\bar{\pi}_2\left(\frac{dy}{d\xi}\right)^2 + \bar{\pi}_4 y^2 + 4\bar{\pi}_3 y\,\frac{dy}{d\xi} + 2\bar{\rho}_2 y\,\frac{d^2y}{d\xi^2} + 4\bar{\pi}_1\,\frac{dy}{d\xi}\cdot\frac{d^2y}{d\xi^2} = 0,$$

where the new coefficients are

$$\bar{\pi}_1 = \tfrac{1}{2}\frac{\mu''}{\mu'^2} + \frac{p_1}{\mu'}, \quad \bar{\pi}_2 = \tfrac{1}{4}\frac{\mu''^2}{\mu'^4} + p_1\frac{\mu''}{\mu'^3} + \frac{p_2}{\mu'^2},$$

$$\bar{\rho}_2 = \frac{q_2}{\mu'^2}, \quad \bar{\pi}_3 = \tfrac{1}{2} q_2 \frac{\mu''}{\mu'^4} + \frac{p_3}{\mu'^3}, \quad \bar{\pi}_4 = \frac{p_4}{\mu'^4}.$$

The infinitesimal transformations are obtained by putting

$$\xi = x - \nu(x)\,\delta t, \tag{16}$$

$\nu(x)$ being an arbitrary function of x, and δt, an infinitesimal. We have written $-\nu(x)$ rather than $+\nu(x)$ so as to harmonize with the infinitesimal transformations of the dependent variable y. If we denote by δx and δy the infinitesimal transformations of x and y respectively, taken both times in the sense—new value of x or y minus the old value—we have, in our notation,

$$\delta x = -\nu(x)\,\delta t, \quad \delta y = -\phi(x)\,\delta t.$$

From (16) follows

$$\left.\begin{aligned} \mu' &= 1 - \nu'\,\delta t, \\ \mu'' &= -\nu''\,\delta t. \end{aligned}\right\} \tag{17}$$

This leads immediately to the following expressions for the infinitesimal changes in the coefficients:

$$\delta p_1 = (\nu' p_1 - \tfrac{1}{2}\nu'')\,\delta t, \quad \delta p_2 = (2\nu' p_2 - p_1\nu'')\,\delta t,$$
$$\delta q_2 = 2\nu' q_2\,\delta t, \quad \delta p_3 = (3\nu' p_3 - \tfrac{1}{2} q_2 \nu'')\,\delta t, \quad \delta p_4 = 4\nu' p_4\,\delta t. \tag{17'}$$

To obtain the variation of a derivative of a function whose variation is known, we make use of the formula

$$\delta f' = \frac{d}{dx}(\delta f) - \frac{df}{dx}\cdot\frac{d}{dx}(\delta x).$$

In the present case, since $\delta x = -\nu\delta t$, this equation becomes

$$\delta f' = \frac{d}{dx}(\delta f) + \nu' f' \delta t. \tag{18}$$

In particular, we have

$$\delta p_1' = (2\nu' p_1' + \nu'' p_1 - \tfrac{1}{2}\nu''')\,\delta t.$$

The variations of the seminvariants can now be obtained. They are given

by the equations

$$\left.\begin{aligned} \delta P_2 &= 2\nu' P_2\,\delta t, \quad \delta Q_2 = (2\nu' Q_2 + \tfrac{1}{2}\nu''')\,\delta t, \\ \delta P_3 &= (3\nu' P_3 + \nu'' P_2)\,\delta t, \\ \delta P_4 &= (4\nu' P_4 + 2\nu'' P_3 + \nu''' Q_2)\,\delta t, \end{aligned}\right\} \tag{19}$$

while $\delta P_2'$, $\delta P_2''$, $\delta Q_2'$, etc., are readily calculated from (19) and (18).

Having thus applied the general infinitesimal transformation of the group to the seminvariants, we may at once write down the equation characteristic of an absolute invariant:

$$\sum_{i=2}^{4}\left(\frac{\partial f}{\partial P_i}\,\delta P_i + \frac{\partial f}{\partial P_i'}\,\delta P_i' + \ldots\right) + \frac{\partial f}{\partial Q_2}\,\delta Q_2 + \frac{\partial f}{\partial Q_2'}\,\delta Q_2' + \ldots = 0. \tag{20}$$

It is easy to verify the fact that for our present equation the following statements, quoted almost verbatim from the paper of Dr. Wilczynski, already referred to, hold true equally as well as for linear equations:

1. Every absolute invariant is isobaric in the coefficients (and therefore in the seminvariants (14)) and of weight zero.

2. An absolute invariant, rational in the seminvariants (14), must be the quotient of two relative invariants of the same weight.

3. A relative invariant is isobaric in the seminvariants (14), and if the common weight of all its terms is w, it satisfies the equation

$$\theta_w(\xi) = (\mu')^{-w}\theta_w(x), \tag{21}$$

or, for infinitesimal transformations,

$$\delta\theta_w = w\theta_w\,\nu'\delta t. \tag{22}$$

For proofs which need scarcely any alteration, Dr. Bouton's paper on the linear equation may be referred to.

The first equation of (19) shows that P_2 is a relative invariant. Therefore,

$$\theta_2 = P_2. \tag{23}$$

Clearly θ_3 must have the form $aP_2' + bQ_2' + cP_3$. Accordingly,

$$\begin{aligned}\delta\theta_3 &= \{a(3\nu' P_2' + 2\nu'' P_2) + b(3\nu' Q_2' + 2\nu'' Q_2 + \tfrac{1}{2}\nu^{(4)}) + c(3\nu' P_3 + \nu'' P_2)\}\,\delta t \\ &= 3\nu'\theta_3\,\delta t,\end{aligned}$$

from which follows $\qquad b = 0, \quad c + 2a = 0,$

so that we have $\qquad \theta_3 = P_3 - \tfrac{1}{2}P_2'. \qquad (24)$

The equation for an absolute invariant involving the seminvariants (14) up to weight 4, breaks up into the following system:

$$\left.\begin{aligned}
&\frac{\partial f}{\partial Q_2''}=0,\quad \frac{\partial f}{\partial Q_2'}=0,\\
&\tfrac{1}{2}\frac{\partial f}{\partial Q_2}+2Q_2\frac{\partial f}{\partial Q_2''}+2P_2\frac{\partial f}{\partial P_2''}+P_2\frac{\partial f}{\partial P_3'}+Q_2\frac{\partial f}{\partial P_4}=0,\\
&2Q_2\frac{\partial f}{\partial Q_2'}+2P_2\frac{\partial f}{\partial P_2'}+P_2\frac{\partial f}{\partial P_3}+5P_2'\frac{\partial f}{\partial P_2''}+5Q_2'\frac{\partial f}{\partial Q_2''}\\
&\qquad\qquad+(3P_3+P_2')\frac{\partial f}{\partial P_3'}+2P_3\frac{\partial f}{\partial P_4}=0,\\
&2P_2\frac{\partial f}{\partial P_2}+2Q_2\frac{\partial f}{\partial Q_2}+3P_2'\frac{\partial f}{\partial P_2'}+3Q_2'\frac{\partial f}{\partial Q_2'}+3P_3\frac{\partial f}{\partial P_3}\\
&\qquad\qquad+4P_2''\frac{\partial f}{\partial P_2''}+4Q_2''\frac{\partial f}{\partial Q_2''}+4P_3'\frac{\partial f}{\partial P_3'}+4P_4\frac{\partial f}{\partial P_4}=0.
\end{aligned}\right\}\quad(25)$$

These are all independent, hence there are $9-5=4$ functionally independent solutions; i. e., four absolute invariants, or five independent relative invariants. These five relative invariants are solutions of the first four equations, their left-hand members having been multiplied into $\nu^{(5)}$, $\nu^{(4)}$, $\nu^{(3)}$, ν'' respectively in (20) (see (22)).

One of these solutions should be the discriminant of the ternary quadratic form corresponding to (2); this is

$$\bar{\theta}_6=P_3^2-P_2(P_4-Q_2^2). \tag{26}$$

Two others we already know, θ_2 and θ_3. Without going through with the process of solving the equations, we know a priori another, given by Forsyth's Jacobian process:*

$$\bar{\bar{\theta}}_6=3\theta_2'\theta_3-2\theta_3'\theta_2. \tag{27}$$

For the fifth, we may take

$$\theta_6=P_3^2+4P_2^2Q_2-2P_2P_3'+2P_2'P_3. \tag{28}$$

These are all independent, as may easily be verified.

Before leaving this part of the subject, let us note an invariant θ_4 which we shall refer to later on. This is connected with the invariants already obtained

* Phil. Trans., I, 1888, pp. 407–418.

by the relation

$$\theta_2\,\theta_4 - 12\theta_3^2 - 3\theta_6 + 15\bar{\theta}_6 - 2\bar{\bar{\theta}}_6 = 0\,.$$

Its value in terms of the seminvariants is

$$\theta_4 = 15\,(P_4 - Q_2^2) - 10P_3' + 2P_2'' + 12P_2\,Q_2. \tag{29}$$

Let us now attempt to find a complete system of independent relative invariants containing seminvariants (14) up to weight σ ($\sigma > 2$). These will be the solutions of the system of equations which (20), taken to terms of weight σ, breaks up into (excluding that one whose left-hand member is multiplied by ν' in (20)). (19) shows us that these equations are σ in number. They are linearly independent; for a consideration of (18) and (19) shows that two of them are

$$\frac{\partial f}{\partial Q_2^{(\sigma-2)}} = 0, \quad \frac{\partial f}{\partial Q_2^{(\sigma-3)}} = 0\,;$$

while if we write down the other equations obtained by equating to zero the coefficients of ν'', $\nu^{(3)}$, ..., in (20) in this order, each contains a term of the form $\varkappa_\rho\,P_2^{(\rho)}\,\dfrac{\partial f}{\partial P_2^{(\sigma-3)}}$ ($\varkappa_\rho$ a constant) which does not occur in any of the preceding equations. Hence, since the arguments are $4\sigma - 7$ in number, there are $3\sigma - 7$ functionally independent solutions.

We may now state

THEOREM II.—*The number of independent relative invariants of* (2) *containing seminvariants* (14) *of weight* σ *or less is* $3\sigma - 7$. ($\sigma > 2$).

It can now be shown that Forsyth's Jacobian process yields, when applied to the invariants already in our possession, a complete system. Let us write

$$\left.\begin{array}{l}
\bar{\bar{\theta}}_6 \;\;\;\; = 3\theta_2'\,\theta_3 - 2\theta_3'\,\theta_2, \\
\theta_9 \;\;\;\; = 6\theta_2'\,\theta_6 - 2\theta_6'\,\theta_2, \\
\bar{\theta}_9 \;\;\;\; = 6\theta_2'\,\bar{\theta}_6 - 2\bar{\theta}_6'\,\theta_2, \\
\bar{\bar{\theta}}_9 \;\;\;\; = 6\theta_2'\,\bar{\bar{\theta}}_6 - 2\bar{\bar{\theta}}_6'\,\theta_2, \\
\;\vdots \qquad\quad \vdots \qquad\qquad \vdots \\
\theta_{3\lambda-6} = (3\lambda - 9)\,\theta_2'\,\theta_{3\lambda-9} - 2\theta_{3\lambda-9}'\,\theta_2, \\
\bar{\theta}_{3\lambda-6} = (3\lambda - 9)\,\theta_2'\,\bar{\theta}_{3\lambda-9} - 2\bar{\theta}_{3\lambda-9}'\,\theta_2, \\
\theta_{3\lambda-6} = (3\lambda - 9)\,\theta_2'\,\bar{\bar{\theta}}_{3\lambda-9} - 2\bar{\bar{\theta}}_{3\lambda-9}'\,\theta_2. \\
\;\vdots \qquad\qquad\quad \vdots \qquad\qquad\qquad \vdots
\end{array}\right\} \tag{30}$$

The invariants θ_2, θ_3, $\theta_{3\lambda-6}$, $\bar{\theta}_{3\lambda-6}$, $\bar{\bar{\theta}}_{3\lambda-6}$, $(\lambda = 4, 5, 6, \ldots, \sigma)$ are $3\sigma - 7$ in number, and they are functionally independent, for, taken in this order, each contains at least one seminvariant not to be found in the preceding; these are successively, P_3, P_3', P_4, P_2'', P_3'', P_4', P_2''', ... , $P_3^{(\sigma-3)}$, $P_4^{(\sigma-4)}$, $P_2^{(\sigma-2)}$. Finally, this set involves seminvariants of weight σ, and of no higher weight; for, by their law of formation, $\theta_{3\lambda-6}$, $\bar{\theta}_{3\lambda-6}$, $\bar{\bar{\theta}}_{3\lambda-6}$ contain seminvariants of weight higher by unity than any entering into $\theta_{3\lambda-9}$, $\bar{\theta}_{3\lambda-9}$, $\bar{\bar{\theta}}_{3\lambda-9}$, and of no higher weight; θ_2 contains a seminvariant of weight 2, θ_3 one of weight 3, while the highest weight of any seminvariant in θ_6, $\bar{\theta}_6$, $\bar{\bar{\theta}}_6$ is 4; a simple induction completes the proof.

We have now established

THEOREM III.—θ_2, θ_3, $\theta_{3\lambda-6}$, $\bar{\theta}_{3\lambda-6}$, $\bar{\bar{\theta}}_{3\lambda-6}$, $(\lambda = 4, 5, \ldots, \sigma)$ *form a complete system of relative invariants containing seminvariants* (14) *of weight* σ *or less* $(\sigma > 2)$; *all other such relative invariants are functionally dependent upon these.*

Hence, all absolute invariants, rational or irrational, containing these seminvariants, depend functionally upon the $3\sigma - 6$ independent rational absolute invariants formed from the above system of relative invariants.

Note that another system might have been obtained by substituting θ_3 for θ_2 in (30).

In passing we may notice another important class of invariant expressions. We have $d\xi = \mu'(x)\,dx$. Accordingly (see (21)), denoting by $\theta_w(\xi)$, as in (21), the invariant corresponding to θ_w formed for the transformed equation,

$$\frac{\theta_{\nu+1}(\xi)}{\theta_\nu(\xi)}\,d\xi = \frac{\mu'^{-(\nu+1)}\,\theta_{\nu+1}(x)}{\mu'^{-\nu}\,\theta_\nu(x)}\,\mu'\,dx = \frac{\theta_{\nu+1}(x)}{\theta_\nu(x)}\,dx.$$

There exists, therefore, a class of integral invariants of the form

$$R_\nu = \int \frac{\theta_{\nu+1}(x)}{\theta_\nu(x)}\,dx. \tag{31}$$

C.—*Semicanonical Form.*

We may choose λ so as to make the coefficient π_1 in the transformed differential equation vanish. Equations (4) show that this may be accomplished by putting

$$\frac{\lambda'}{\lambda} = -p_1,$$

or

$$\lambda = Ce^{-\int p_1\,dx}.$$

As the expressions for the seminvariants show, and as may be verified by substituting the above value for λ into equations (4), the transformed equation becomes

$$\eta''^2 + 4P_2\eta'^2 + P_4\eta^2 + 4P_3\eta\eta' + 2Q_2\eta\eta'' = 0. \tag{32}$$

Its coefficients are seminvariants. This form of the differential equation, which is characterized by the condition $p_1 = 0$, will be called the semicanonical form. The transformation leading to it requires only a quadrature.

D.—*Canonical Form.*

The general transformation (1) contains two arbitrary functions, λ and μ, so that we should be able to obtain a transformed equation lacking any two terms, except of course the first. But, in general, the equations determining the values of λ and μ for such a transformation are not solvable by quadrature alone. We can, however, by mere quadratures, determine λ and μ so as to make the coefficients of yy', $y'y''$ disappear in the transformed equation; i. e., we can reduce (2) to the form,

$$\left(\frac{d^2Y}{dX^2}\right)^2 + 4I\left(\frac{dY}{dX}\right)^2 + JY^2 + 2KY\frac{d^2Y}{dX^2} = 0, \tag{33}$$

which we shall call the canonical form.

Let the transformation reducing (2) to this form be

$$\left.\begin{aligned} y &= \Lambda(x)\,Y, \\ X &= \mathrm{M}(x). \end{aligned}\right\} \tag{34}$$

Instead of actually carrying out such a transformation, we may determine I, J and K by means of the invariants of (2).

We find that for equation (33),

$$\theta_2 = I, \quad \theta_3 = -\tfrac{1}{2}\frac{dI}{dX} = -\tfrac{1}{2}\frac{dI}{dx}\,\frac{dx}{dX},$$
$$\theta_6 = 4I^2K, \quad \bar{\theta}_6 = -I(J-K^2).$$

Therefore, making use of (21), we have

$$\left.\begin{aligned} I &= \frac{\theta_2}{M'^2}, \quad \frac{-I'}{2M'} = \frac{\theta_3}{M'^3}, \\ 4I^2K &= \frac{\theta_6}{M'^6}, \quad -I(J-K^2) = \frac{\bar{\theta}_6}{M'^6}. \end{aligned}\right\} \tag{34'}$$

From these equations we can easily calculate I, J, K, their values being

$$I = \frac{\theta_2}{M'^2}, \quad J = \frac{\theta_6^2 - 16\theta_2^3\bar{\theta}_6}{16\theta_2^4 M'^4}, \quad K = \frac{\theta_6}{4\theta_2^2 M'^2}. \tag{35}$$

In addition, we obtain from (34′)

$$\left.\begin{aligned} M'' &= \left(\frac{\theta_3}{\theta_2} + \frac{1}{2}\frac{\theta_2'}{\theta_2}\right) M', \\ M' &= C_2\, \theta_2^{\frac{1}{2}} e^{\int_c^x \frac{\theta_3}{\theta_2} dx}, \\ M &= C_2 \int_c^x \theta_2^{\frac{1}{2}} e^{\int_c^x \frac{\theta_3}{\theta_2} dx} dx + C_3, \end{aligned}\right\} \tag{36}$$

C_2 and C_3 being arbitrary constants.

If we apply (34) to (2), the coefficient of $Y'Y''$ in the transformed equation will be

$$\frac{1}{2}\frac{M''}{M'^2} + \frac{1}{M'}\left(\frac{\Lambda'}{\Lambda} + p_1\right).$$

Since this must vanish, we have the following determination for $\Lambda(x)$:

$$\Lambda(x) = C_1\, \theta_2^{-\frac{1}{4}} e^{-\int_c^x \left(p_1 + \frac{1}{2}\frac{\theta_3}{\theta_2}\right) dx} \tag{37}$$

We have, in the preceding work, taken it for granted that a transformation exists which will reduce to form (33) equation (2). We may, however, by actual substitution, readily verify the fact that (34), as determined by (36) and (37), really does reduce (2) to the desired form.

We should note one exceptional case where the canonical form fails, namely, where $\theta_2 = 0$. An extended discussion of this case will not be attempted in this paper. We may here reduce (2) to a form containing only three terms, but the transformation can no longer be obtained, in general, by mere quadratures. In fact, it may be readily verified that if ξ is any solution of the equation

$$\xi'' + Q_2 \xi = 0,$$

the transformation

$$\left\{\begin{aligned} y &= e^{-\int_c^x p_1 dx} \xi Y, \\ X &= k \int_c^x \frac{dx}{\xi^2}, \end{aligned}\right.$$

where k is any constant, reduces (2) to the form

$$\left(\frac{d^2Y}{dX^2}\right)^2 + \frac{\xi^6\theta_3}{k^3}\,Y\frac{dY}{dX} + \frac{\xi^8(\theta_4 + 10\theta_3') + 60\theta_3\xi^7\xi'}{15k^4}\,Y^2 = 0. \tag{38}$$

At this point we may compare Appell's results. (33) is substantially his canonical form, but throughout he uses only indefinite integrals, neglecting the constants in (36) and (37); I, J, K and $\int M'\,dx$ are then absolute invariants, and these, with the derivatives of I, J, K, with regard to X, form a complete system. To show how this system may be connected with the one we have assigned, the following equations may serve as examples:

$$\frac{4K}{I} = \frac{\theta_6}{\theta_2^3}, \quad \frac{K^2 - J}{I^2} = \frac{\bar{\theta}_6}{\theta_2^3},$$

$$\frac{\left(\dfrac{dI}{dX}\right)^2}{4I^3} = \frac{\theta_3^2}{\theta_2^3}.$$

Our system has the advantage of giving us explicitly rational integral relative invariants, and also takes into account the constants of (36) and (37), thus giving the most general transformation reducing (2) to the canonical form. To find the most general transformation leaving (33) invariant, we need only substitute M'^2I for θ_2 and $-\frac{1}{2}M'^2I'$ for θ_3 in (36) and (37). The result is

$$y = C_1Y, \quad X = C_2x + C_3, \tag{39}$$

The most general infinitesimal transformation leaving the semicanonical form invariant is easily deduced. The total infinitesimal change in a coefficient for transformations (5) and (16) together is evidently the sum of the changes due to the transformation of each variable separately. We wish, then, to obtain all those transformations which make δp_1 vanish if p_1 is zero. From (6) and (17′) we find

$$\delta p_1 = (\phi' + \nu' p_1 - \tfrac{1}{2}\nu'')\,\delta t.$$

Hence for any infinitesimal transformation leaving the semicanonical form unchanged,

$$\phi' - \tfrac{1}{2}\nu'' = 0;\ \text{i. e.,}\ \phi = \tfrac{1}{2}\nu' + c \qquad (c \text{ an arbitrary constant}).$$

This gives the infinite subgroup, whose infinitesimal transformations are

$$\left.\begin{aligned}\delta y &= -\phi(x)\, y\,\delta t = -\left(\tfrac{1}{2}\nu'(x) + c\right) y\,\delta t,\\ \delta x &= -\nu(x)\,\delta t.\end{aligned}\right\} \tag{40}$$

The corresponding finite transformations are

$$\left.\begin{aligned}y &= k\mu'^{-\frac{1}{2}}\eta \qquad (k \text{ an arbitrary constant}),\\ \xi &= \mu(x).\end{aligned}\right\} \tag{41}$$

The equations for the most general subgroup leaving the canonical form invariant are

$$\phi' - \tfrac{1}{2}\nu'' = 0, \quad (2p_2 + q_2)\phi' - \tfrac{1}{2}q_2\nu'' = 0,$$

giving

$$\phi = k_1, \quad \nu = k_2 x + k_3.$$

The infinitesimal transformations sought for are therefore

$$\left.\begin{aligned}\delta y &= -k_1 y\,\delta t,\\ \delta x &= -(k_2 x + k_3)\,\delta t,\end{aligned}\right\} \tag{42}$$

and, of course, (39) gives the corresponding finite transformations. We should note that under the subgroup (39) the derivative of a relative invariant is itself invariant. For, by (18) and (22),

$$\delta\theta'_\sigma = \frac{d}{dx}(\delta\theta_\sigma) + \nu'\,\theta'_\sigma\,\delta t = \{\sigma\nu''\theta_\sigma + (\sigma+1)\nu'\theta'_\sigma\}\,\delta t.$$

But, since for all infinitesimal transformations of the subgroup $\nu'' = 0$,

$$\delta\theta'_\sigma = (\sigma+1)\,\nu'\,\theta'_\sigma\,\delta t;$$

i. e., θ'_σ is an invariant of the subgroup. Clearly any transformation (2) agreeing with (39) in its change of independent variable has the same property.

E.—*Remarks and Applications.*

If θ_2 vanishes identically, the complete system of Theorem III apparently reduces to θ_3 alone, while if both θ_2 and θ_3 vanish, every member of the system become equal to zero. The vanishing of θ_2 and θ_3 means no more or less than the vanishing of P_2 and P_3; hence we might go back to the differential equation (20), omitting terms in P_2, P_3 and their derivatives, and from its solutions build up a new complete system. Nevertheless, the system of Theorem III, though all its members vanish, is still complete; all relative invariants may be obtained

as the *limits* of combinations of its members; an illustration is afforded by θ_4 which equals $15(P_4 - Q_2^2)$ when $\theta_2 = \theta_3 = 0$.

For some purposes a form of (2), in which $(y'')^2$ has a coefficient p_0 different from unity, may be advantageous; it will be general enough to take p_0 as a constant. In this case p_0 is absolutely invariant. Seminvariants and relative invariants are readily obtained from those given in this paper by writing $\frac{p_1}{p_0}$, $\frac{p_2}{p_0}$, etc., for p_1, p_2, etc., and multiplying the resulting forms by a power of p_0 sufficient to clear of fractions. Isobaric seminvariants and invariants are then homogeneous in the coefficients. From these results we could easily construct the invariants of a form of (2) lacking the term in $(y'')^2$.

In conclusion, a few applications are here given, the discussion in each case being as brief as is consistent with clearness.

1. Two equations of form (2) are equivalent under a transformation (3) of the dependent variable alone, when all the seminvariants of the one are the same functions of the independent variable as are those of the other. This condition is necessary, as is clear from the definition of a seminvariant, and it is also sufficient, since the semicanonical forms of the two equations are then identical.

2. Two equations of form (2) for which $\theta_2 \neq 0$ are equivalent under a transformation of the group (1) if a transformation $\xi = \bar{\mu}(x)$ exists which, when applied to the absolute invariants entering into the canonical form of one equation, gives the corresponding invariants of the other equation.

First, this is necessary; for, suppose a transformation $\xi = \bar{\mu}(x)$, $y = \bar{\lambda}(x)\eta$ changes an equation (A) in terms of x and y into an equation (B) in terms of ξ and η. The absolute invariants of (B) are in terms of ξ, and, from the nature of an invariant, must *all* be identical with the corresponding invariants of (A) if $\bar{\mu}(x)$ be substituted for ξ.

This condition is also sufficient; for transform (A) into an equation (C) in ξ and y by means of the substitution $\xi = \bar{\mu}(x)$. The invariants of (C) are now identical with those of (B); and we may easily show that (C) and (B) are equivalent under a transformation $y = \bar{\lambda}(x)\eta = \lambda(\xi)\eta$ of the dependent variable alone. To prove this, reduce both to the canonical form. Now $\frac{\theta_3}{\theta_2}\, d\xi$ and $\theta_2^{\frac{1}{2}} d\xi$ are both absolute invariants; this may be shown as in the work preceding (31); hence, choosing C_2, C_3, and c the same for each (see (36)), M is identically the same

for (C) as for (B). I, J, K are the products of absolute invariants and expressions of the form

$$\left(\frac{1}{C_2 e^{\int_c^\xi \frac{\theta_3}{\theta_2} d\xi}}\right)^r,$$

r being either 2 or 4, and are therefore also the same for (C) and (B). Thus (C) and (B) are reduced to identically the same equation by transformations which agree so far as the independent variable is concerned; they are equivalent under a transformation of the dependent variable alone; (A) is transformed into (B) by the substitution of variables $\xi = \bar{\mu}(x)$, $y = \bar{\lambda}(x)\eta$.

To apply this test to two equations in x and ξ, we should equate the absolute invariants entering into the canonical form of each; if these relations all give the same solution for ξ in terms of x, the two equations are equivalent. In particular, if one has constant coefficients, the absolute invariants of the other must reduce to constants.

Note that the preceding work shows that if the absolute invariants entering into the canonical form of the one are carried into the corresponding invariants of the other by the transformation $\xi = \bar{\mu}(x)$, the equations are equivalent, and, therefore, this same transformation carries *all* the absolute invariants of the one into those of the other. Thus the invariants of an equation (2) are completely determined by those which enter into the canonical form.

Remembering that one invariant entering into the canonical form is $\frac{\theta_3}{\theta_2}\,dx$, it will at once be seen that the equivalence condition may be given in the following form:

$$\theta_2(\xi) = \left(\frac{d\xi}{dx}\right)^{-2}\theta_2(x), \quad \theta_3(\xi) = \left(\frac{d\xi}{dx}\right)^{-3}\theta_3(x),$$
$$\theta_6(\xi) = \left(\frac{d\xi}{dx}\right)^{-6}\theta_6(x), \quad \bar{\theta}_6(\xi) = \left(\frac{d\xi}{dx}\right)^{-6}\bar{\theta}_6(x),$$

the invariants on the left-hand side of each equation being formed for the equation in ξ, those on the right for the equation in x.

We have assumed here that $\theta_2 \neq 0$, i. e., that a canonical form is possible; we shall not in this paper attempt the discussion of the case $\theta_2 = 0$.

3. If $\bar{\theta}_6$ vanishes identically, (2) breaks up into two linear equations of the second order.

4. If $\theta_2 = \theta_3 = \theta_4 = 0$, (2) is the square of a linear equation of the second order; (2) then has no invariants. For if $\theta_2 = \theta_3 = \theta_4 = 0$,

$$P_2 = P_3 = 0,$$
$$P_4 - Q_2^2 = 0;$$

the form (38) reduces to

$$(Y'')^2 = 0.$$

Form (38) also gives us a binomial form

$$\left(\frac{d^2Y}{dX^2}\right)^2 + \frac{\xi^8\theta_4}{15K^4}\,Y^2 = 0,$$

to which (2) is reducible in case $\theta_2 = \theta_3 = 0$.

5. Appell, in the article already referred to, has developed the condition that (2) should have for its general solution

$$y = h^2u_1 + hku_2 + k^2u_3,$$

h and k being arbitrary constants, and u_1, u_2, u_3 linearly independent solutions. The conditions he develops are $K = C$, $J = C^2 + 4CI$, C being a constant. One of these conditions may be replaced by the relation $\theta_6 + \bar{\theta}_6 = 0$.

UNIVERSITY OF CALIFORNIA, BERKELEY.